Math 151
Section 2.4

Chain Rule

The chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 1

$$y = \left(3x^2 + 5\right)^5$$
$$Let\ \ y = u^5 \ \ where\ u = 3x^2 + 5 \Rightarrow du = 6x$$
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$y' = 5u^4 du$$
$$y' = 5(3x^2 + 5x)^4 (6x)$$
$$y' = (30x)(3x^2 + 5x)^4$$

Example 2

$$y = \left(x^2 + 3x\right)^7$$
$$Let\ \ y = u^7 \ \ where\ u = x^2 + 3x \Rightarrow du = 2x + 3$$
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$y' = 7u^{7-1} du$$
$$y' = 7u^6 du$$
$$y' = 7(x^2 + 7x)^6 (2x + 3)$$
$$y' = (14x + 21)(x^2 + 3x)^6$$

Example 3

Find the derivative of $y = \sqrt{x^2 - 6x}$

$$y = \sqrt{x^2 - 6x}$$
$$y = (x^2 - 6x)^{\frac{1}{2}}$$
$$Let\ \ y = u^{\frac{1}{2}}\ \ where\ u = x^2 - 6x \Rightarrow du = 2x - 6$$
$$y' = \frac{1}{2}u^{\frac{1}{2}-1}du$$
$$y' = \frac{1}{2}u^{-\frac{1}{2}}du$$
$$y' = \frac{1}{2}(x^2 - 6x)^{-\frac{1}{2}}(2x - 6)$$
$$y' = \frac{2x - 6}{2(x^2 - 6x)^{\frac{1}{2}}}$$
$$y' = \frac{2x - 6}{\sqrt{x^2 - 6x}}$$

Example 4

Find the derivative of $f(x) = \sqrt{x^2 - 6x}$

$$y = \sqrt{x^3 - 3x}$$
$$y = (x^3 - 3x)^{\frac{1}{2}}$$
$$Let\ \ y = u^{\frac{1}{2}}\ \ where\ u = x^3 - 3x \Rightarrow du = 3x^2 - 3$$
$$y = \frac{1}{2}u^{\frac{1}{2}-1}du = \frac{1}{2}u^{-\frac{1}{2}}du = \frac{1}{2}(x^3 - 3x)^{-\frac{1}{2}}(3x^2 - 3) = \frac{3x^2 - 3}{2(x^3 - 3x)^{\frac{1}{2}}} = \frac{3x^2 - 3}{2\sqrt{x^3 - 3x}}$$

Example 5

$$y = \sqrt[3]{x^2 + 4}$$

$$y = \left(x^2 + 4\right)^{\frac{1}{3}}$$

$$Let \quad y = u^{\frac{1}{3}} \; where \; u = x^2 + 4 \Rightarrow u = 2x$$

$$y' = \frac{1}{3} u^{\frac{1}{3}-1} du$$

$$y' = \frac{1}{3} u^{-\frac{2}{3}} du$$

$$y' = \frac{du}{3u^{\frac{2}{3}}}$$

$$y' = \frac{2x}{3(x^2 + 4)^{\frac{2}{3}}}$$

$$y' = \frac{2x}{3\sqrt[2]{x^2 + 4}}$$

Example 6

Find the derivative of $y = \cos(5x)$

$$y = \cos(5x)$$
$$u = 5x$$
$$du = 5$$
$$y = \cos u$$
$$y' = \cos u \cdot du$$
$$y' = 5\cos(5x)$$

Example 7

Find the $y = e^{3x^2}$

$y = e^{3x^2}$

$u = 3x^2$
$du = 6x$

$y' = e^u du$
$y' = e^{3x^2}(6x)$
$y' = 6xe^{3x^2}$

Example 8

Find the $y = e^{x^2 - 6x}$

$y = e^{x^2 - 6x}$

$u = x^2 - 6x$
$du = 2x - 6$

$y' = e^u du$
$y' = e^{x^2 - 6x}(2x - 6)$
$y' = (2x - 6)e^{x^2 - 6x}$

Example 9

Find the derivative of $y = \cos(5x)$

$$y = e^{3x}\cos(6x)$$

$$y' = \frac{d}{dx}e^{3x}\cos(6x) + \frac{d}{dx}(\cos(6x))e^{3x}$$
$$y' = 3e^{3x}\cos(6x) - 6e^{3x}\sin(x)$$
$$y' = 3e^{3x}(\cos(6x) - 2\sin(6x))$$

Example 10

Find the derivative of $y = 4xe^{x^2}$

$$y = 4xe^{x^2}$$
$$y' = \frac{d}{dx}(4x)e^{x^2} + \frac{d}{dx}e^{x^2}(4x)$$
$$y' = 4e^{x^2} + (4x)(2x)e^{x^2}$$
$$y = 4e^{x^2} + 8x^2e^{x^2}$$
$$y' = \left(4 + 8x^2\right)e^{x^2}$$

The derivative of the natural logarithm function

$$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$$

Example 11

Find the derivative of $f(x) = 4\ln(x)$

$$f'(x) = 4\left(\frac{1}{x}\right) = \frac{4}{x}$$

Chain Rule for the natural logarithm

$$\frac{d}{dx}\left(\ln u\right) = \frac{1}{u}du = \frac{du}{u}$$

Example 12

Find the derivative of $f(x) = \ln(3x^3)$

$$Let\ \ f(x) = \ln u\ \ where\ u = 3x^2 \quad \Rightarrow du = 6x$$
$$f'(x) = \frac{du}{u} = \frac{6x}{3x^2} = \frac{2}{x}$$

Example 13

Find the derivative of $f(x) = e^{4x^2}\sin(3x)$

$$f'(x) = \left(e^{4x^2}\right)'\left(\sin(3x)\right) + \left(\sin(3x)\right)'\left(e^{4x^2}\right)$$
$$f'(x) = 8xe^{4x^2}\sin(3x) + \left(3\cos(3x)\right)e^{4x^2}$$
$$f(x) = 8xe^{4x^2}\sin(3x) + 3e^{4x^2}\cos(3x)$$

Example 14

Find the derivative of $y = x^3\ln(5x)$

$$y' = \left(x^3\right)'\ln(5x) + \left(\ln(5x)\right)'\left(x^3\right)$$
$$y' = 3x^2\ln(5x) + \frac{5}{5x}\cdot x^3$$
$$y' = 3x^2\ln(5x) + x^2$$

Example 15

Find the slope of tangent to the function $y = \left(x^2 + 2x\right)^3$ at the point (1,27)

$$y = \left(x^2 + 2x\right)^3$$
$$Let\ \ y = u^3\ \ where\ u = x^2 + 2x \Rightarrow du = 2x + 2$$
$$y' = 3u^{3-1}du$$
$$y' = 3u^2 du$$
$$y' = 3(x^2 + 2x)^2(2x + 2)$$
$$y' = (6x + 6)(x^2 + 3x)^2$$

Slope

$$y' = (6(1) + 6)(1^2 + 2(1))^2 = (6 + 6)(1 + 2)^2 = 12(3)^2 = 12(9) = 108$$
$$m = 108$$

Example 16

The equation of motion of a particle is $s(t) = t^3 - 6t$ where s is in meters and t is seconds.

a) Find the velocity and acceleration as a function of t.

$$v(t) = s'(t) = 3t^{3-1} - 6 = 3t^2 - 6$$
$$a(t) = s''(t) = 6t$$

b) Find the velocity after 2 seconds.

$$v(t) = 3t^{3-1} - 6 = 3t^2 - 6$$
$$v(2) = 3(2)^2 - 6 = 12 - 6 = 4\ \frac{m}{s}$$

c) Find the acceleration after 2 seconds.

$$a(t) = s''(t) = 6t$$
$$a(2) = 6(2) = 12\ \frac{m}{s^2}$$

Example 17

The equation of motion of a particle is $s(t) = 2t^3 - 12t$ where s is in meters and t is seconds.

a) Find the velocity and acceleration as a function of t.

$$v(t) = s'(t) = 3 \cdot 2t^{3-1} - 12 = 6t^2 - 12$$
$$a(t) = s''(t) = 12t$$

b) Find the velocity after 3 seconds.

$$v(t) = 6t^2 - 12$$
$$v(3) = 6(3)^2 - 12 = 54 - 12 = 42\ \frac{m}{s}$$

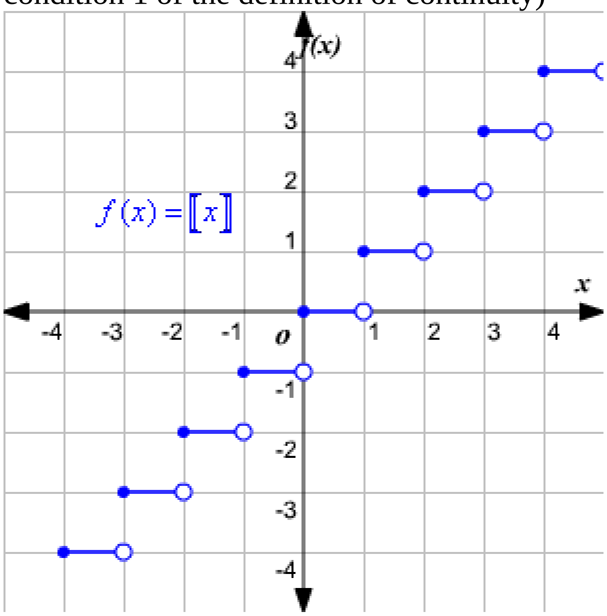
c) Find the acceleration after 3 seconds.

$$a(t) = s''(t) = 12t$$
$$a(3) = 12(3) = 36\ \frac{m}{s^2}$$

Differentiability
A function is **differentiable** at a value x, if its derivative exists at x.
A function is **not differentiable** a value x, if its derivative does not exist at x.
A function is not differentiable in the following situations.

1) A value where the limit doesn't exist.

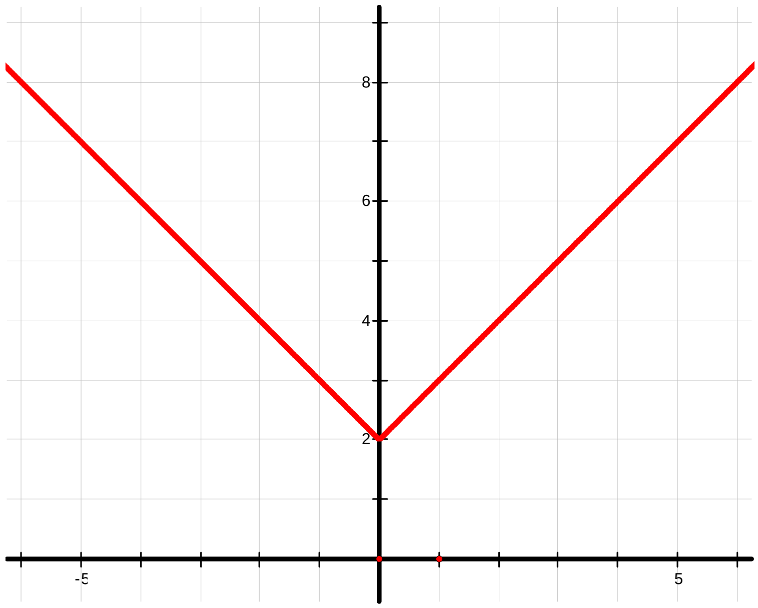
The function below does not have a limit at $x = 1$
$\lim_{x \to 1^+} f(x) = 1$ and $\lim_{x \to 1^-} f(x) = 0$
Therefore, the limit does not exist because the left hand limit and right hand limit are different are different.
Since the limit doesn't exist at $x = 1$, the function is discontinuous at $x = 1$. (See condition 1 of the definition of continuity)



Graph courtesy http://hotmath.com/hotmath_help/topics/step-function.html

2) At a sharp edge in the graph of the function.

$f(x) = |x| + 2$ at $x = 0$



This function is not differentiable at $x = 0$, because the derivative is different on both sides of $x = 0$

Right Hand Derivative:

$$f(x) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{|x| + 2 - (|0| + 2)}{x} = \lim_{x \to 0^+} \frac{|x| + 2 - 2}{x} = \lim_{x \to 0^+} \frac{|x|}{x} = 1$$

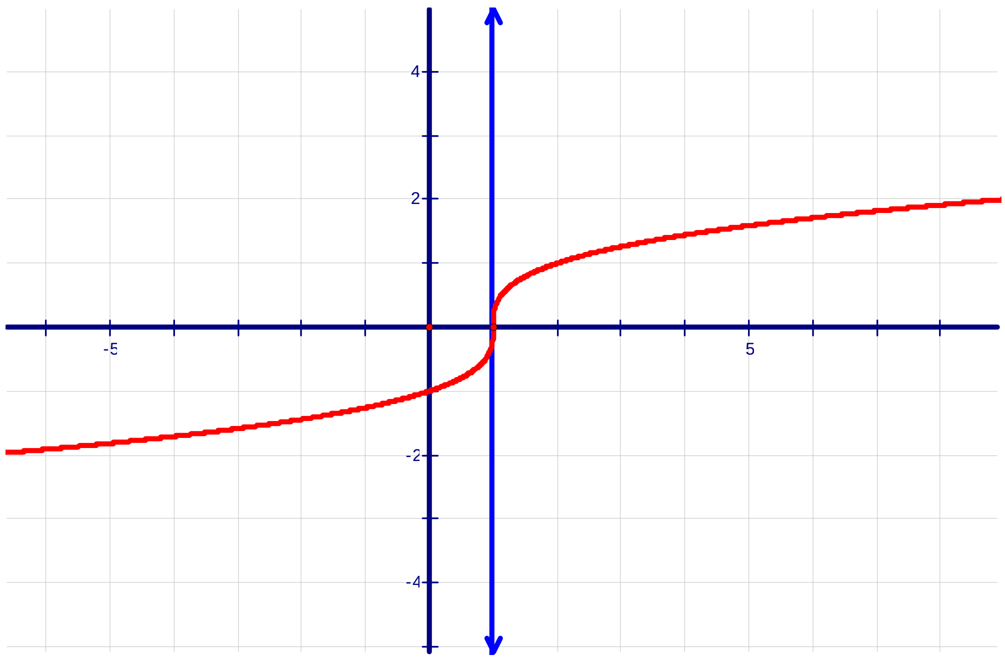
Left Hand Derivative:

$$f(x) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{|x| + 2 - (|0| + 2)}{x} = \lim_{x \to 0^-} \frac{|x| + 2 - 2}{x} = \lim_{x \to 0^-} \frac{|x|}{x} = -1$$

If you look at the slope of the tangent on the right hand side it is 1, where as the slope of the tangent line is – 1 on the left hand side. Therefore, the derivative does not exist at $x = 0$

3) A vertical tangent line to the graph of the function.

$f(x) = \sqrt[3]{x-1}$ at $x = 1$



The slope of the tangent line at $x = 1$ is undefined because we have a vertical tangent line at the value $x = 1$. The derivative is also undefined at $x = 1$

$$f(x) = \sqrt[3]{x-1} \Rightarrow f(x) = (x-1)^{\frac{1}{3}}$$

$$u = x - 1$$

$$\frac{du}{dx} = 1$$

$$f(x) = u^{\frac{1}{3}}$$

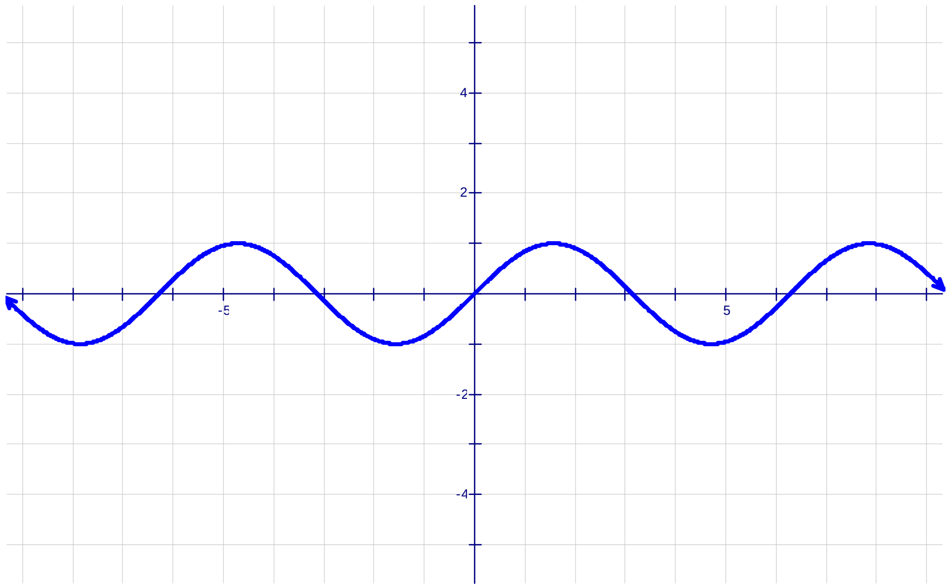
$$f(x) = \frac{1}{3}u^{-\frac{2}{3}}\frac{du}{dx} = \frac{1}{3}(x-1)^{-\frac{2}{3}}(1) = \frac{1}{3\sqrt[3]{(x-1)^2}}$$

$$f'(1) = \frac{1}{3\sqrt[3]{(1-1)^2}} = \frac{1}{3\sqrt[3]{0}} = \frac{1}{0}$$

Example 1

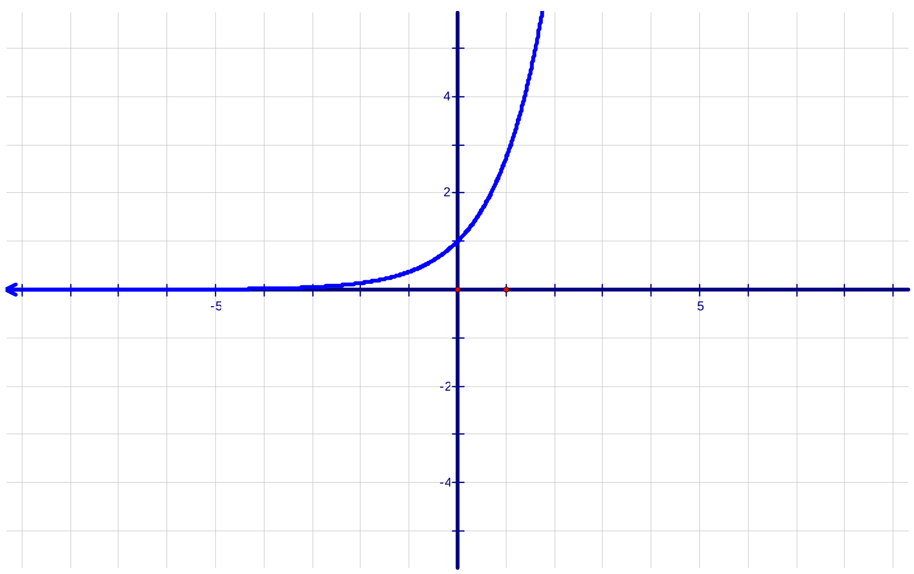
Use the graph of the function to determine what values of x the function is differentiable.

1) $f(x) = \sin x$



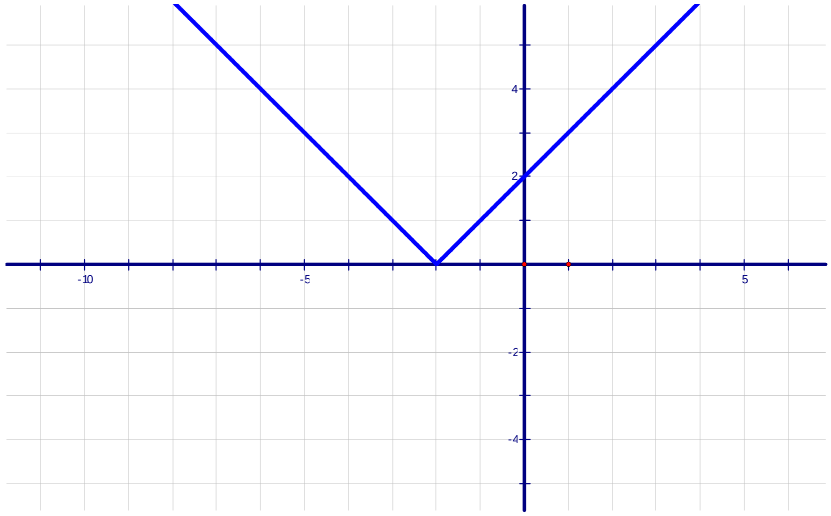
Since the graph has no sharp edges or vertical tangents the function is differentiable for all values of x.

2) $f(x) = e^{x+1}$



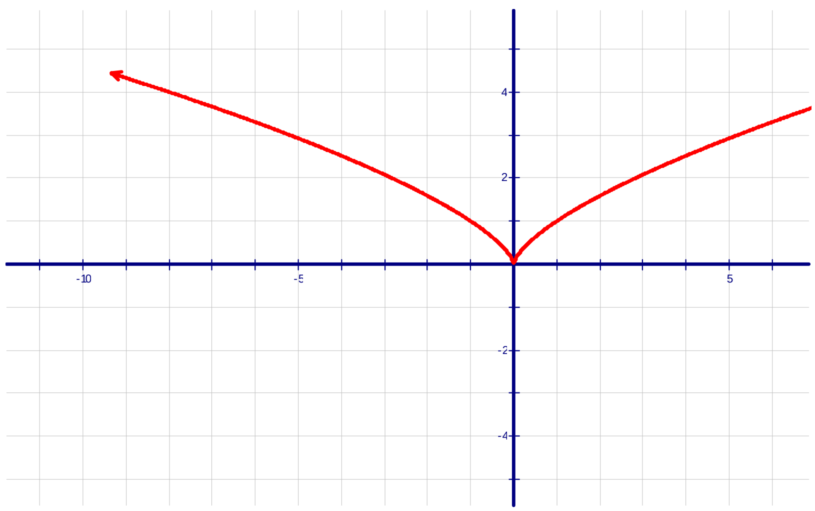
Since the graph has no sharp edges or vertical tangents the function is differentiable for all values of x.

3) $f(x) = |x+2|$



This graph has a sharp edge at $x = -2$. Thus, the function is not differentiable at $x = -2$

4) $f(x) = x^{\frac{2}{3}}$



The graph has a vertical tangent line at x = 0 Also , note that the derivative does exist at x = 0

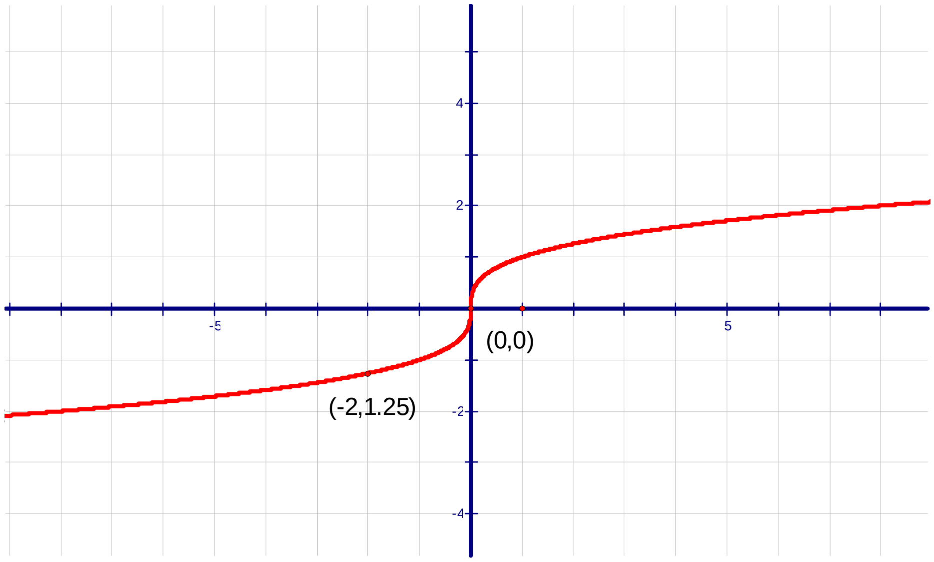
$f(x) = x^{\frac{2}{3}}$

$\Rightarrow \dot{f}'(x) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$

$\Rightarrow f'(0) = \frac{2}{3\sqrt[3]{0}} = \frac{2}{0}$

Example 2

Determine if $f(x) = \sqrt[3]{x}$ is differentiable at $x = -2$ and $x = 0$



The function is differentiable at $x = -2$
The function is not differentiable at $x = 0$ (The function has a vertical tangent at $x = 0$)